H2 Mathematics (9758)

Graphs & Transformations — Lecture Notes

A-Level 2027 Syllabus | Chapter 2

Graph Characteristics

Syllabus Learning Outcomes

By the end of this topic, you should be able to:

- Use a GC to sketch and analyse graphs of given functions
- Identify and interpret: **asymptotes**, **intercepts**, **turning points**, **symmetry**, and restrictions on x and y
- Sketch graphs of conic sections ($y^2 = ax$, $x^2 = by$, ellipses, hyperbolas)
- Sketch graphs of rational functions (linear/linear and quadratic/linear)
- Apply transformations to graphs of $y = f(x)$ and sketch the results
- Understand and sketch $y = |f(x)|$, $y = f(|x|)$, and $y = 1/f(x)$
- Sketch graphs from parametric equations

The Anatomy of a Graph

Every graph you sketch for A-Level must show, where applicable:

Definition Essential Graph Features

1. **x-intercepts** — where $y = 0$. Solve $f(x) = 0$.
2. **y-intercept** — where $x = 0$. Evaluate $f(0)$.
3. **Asymptotes** — lines the graph approaches but never touches.
   - **Vertical asymptotes:** denominator $= 0$, numerator $\neq 0$
   - **Horizontal asymptotes:** behaviour as $x \to \pm\infty$
   - **Oblique asymptotes:** when degree(numerator) $>$ degree(denominator) by exactly 1
4. **Turning points** — where $f'(x) = 0$ (local maxima/minima)
5. **Axes of symmetry** — lines about which the graph is symmetric
6. **Restrictions** on x and/or y — domain and range constraints

Exam Tip The sketching checklist

Before submitting any sketch, verify: asymptotes labelled (as dashed lines), intercepts labelled with coordinates, axes labelled x and y, scale indicated. A sketch missing any of these loses marks.

Conic Sections

Parabolas: $y^2 = ax$ and $x^2 = by$

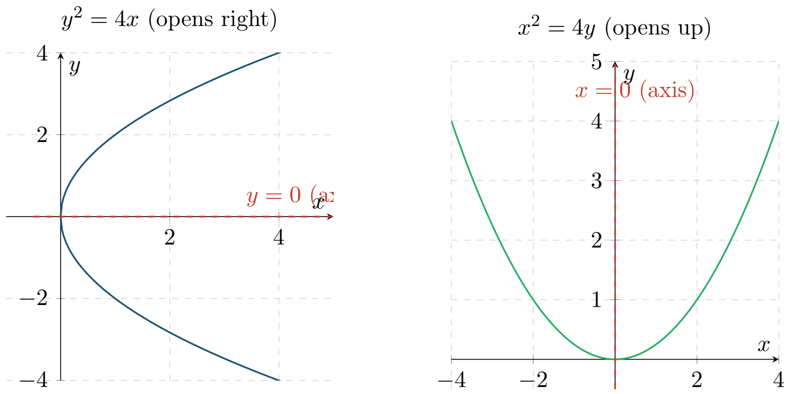
These are "sideways" parabolas. Unlike $y = ax^2 + bx + c$ which is a function, $y^2 = ax$ is **not a function** (it fails the vertical line test).

Definition Parabola $y^2 = ax$

For $a > 0$: opens to the right. Vertex at $(0, 0)$. Axis of symmetry is the x-axis ($y = 0$).
Domain: $x \geq 0$ if $a > 0$; $x \leq 0$ if $a < 0$.
Range: all real y.

Definition Parabola $x^2 = by$

For $b > 0$: opens upward. Vertex at $(0, 0)$. Axis of symmetry is the y-axis ($x = 0$).
Domain: all real x. Range: $y \geq 0$ if $b > 0$; $y \leq 0$ if $b < 0$.



Ellipses: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Definition Ellipse

Standard form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Centre at $(0, 0)$. Semi-major axis: $\max(a, b)$. Semi-minor axis: $\min(a, b)$.
If $a > b$: major axis is horizontal (along x-axis).
If $b > a$: major axis is vertical (along y-axis).
Intercepts: $(\pm a, 0)$ and $(0, \pm b)$.
Domain: $-a \leq x \leq a$. Range: $-b \leq y \leq b$.

Hyperbolas

Definition Hyperbola — opening left/right

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Centre at $(0, 0)$. Opens along the x-axis.
x-intercepts: $(\pm a, 0)$. No y-intercepts.
Asymptotes: $y = \pm\frac{b}{a}x$.
Domain: $x \leq -a$ or $x \geq a$. Range: all real y.

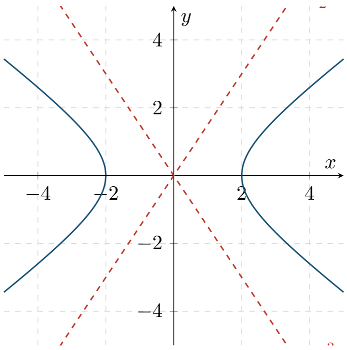
Definition Hyperbola — opening up/down

$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$
Opens along the y-axis.
y-intercepts: $(0, \pm b)$. No x-intercepts.
Asymptotes: $y = \pm\frac{b}{a}x$ (same!).
Range: $y \leq -b$ or $y \geq b$. Domain: all real x.

Warning Hyperbola direction

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has the x^2 term positive — opens along x-axis.
$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ has the y^2 term positive — opens along y-axis.

$\frac{x^2}{4} - \frac{y^2}{9} = 1$



Rational Functions

The Reciprocal Function $y = \frac{ax+b}{cx+d}$

This is the most fundamental rational function tested.

Definition Reciprocal / Linear-over-Linear

$y = \frac{ax+b}{cx+d}$, where $c \neq 0$ and $ad - bc \neq 0$ (otherwise it simplifies to a constant).
Vertical asymptote: $x = -\frac{d}{c}$ (denominator $= 0$)
Horizontal asymptote: $y = \frac{a}{c}$ (limit as $x \to \pm\infty$)
Centre: $\left(-\frac{d}{c}, \frac{a}{c}\right)$ — the intersection of asymptotes.
A rectangular hyperbola, symmetric about $y = x$ and $y = -x$ through the centre.

Derivation Finding the horizontal asymptote

$$\lim_{x\to\infty} \frac{ax+b}{cx+d} = \lim_{x\to\infty} \frac{a+b/x}{c+d/x} = \frac{a}{c}$$

Since $\frac{b}{x} \to 0$ and $\frac{d}{x} \to 0$ as $x \to \infty$.

Example Sketching a rational function

Sketch $y = \frac{2x+1}{x-3}$.
Solution:

- **VA:** $x - 3 = 0 \Rightarrow x = 3$
- **HA:** $y = \frac{2}{1} = 2$
- **x-intercept:** $y = 0 \Rightarrow 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$
- **y-intercept:** $x = 0 \Rightarrow y = \frac{1}{-3} = -\frac{1}{3}$
- **Centre:** $(3, 2)$

Now test one point on each side of the VA to determine the branches' behaviour.
At $x = 4$: $y = \frac{9}{1} = 9$ (above HA, right branch goes up)
At $x = 2$: $y = \frac{5}{-1} = -5$ (below HA, left branch goes down)

Quadratic-over-Linear: $y = \frac{ax^2+bx+c}{dx+e}$

These have **oblique asymptotes** (also called slant asymptotes) because the numerator's degree exceeds the denominator's by 1.

Definition Oblique Asymptote

When degree(numerator) = degree(denominator) + 1, perform long division:

$$\frac{ax^2 + bx + c}{dx + e} = (\text{quotient}) + \frac{\text{remainder}}{dx + e}$$

The quotient (a linear expression) is the **oblique asymptote**. The graph approaches this line as $x \to \pm\infty$.

Example Oblique asymptote

Find the asymptotes of $y = \frac{x^2+2x+3}{x-1}$.
Solution:
Long division: $x^2 + 2x + 3 \div (x - 1)$
$x^2 \div x = x$, multiply: $x(x - 1) = x^2 - x$, subtract: $3x + 3$
$3x \div x = 3$, multiply: $3(x - 1) = 3x - 3$, subtract: 6
So $y = x + 3 + \frac{6}{x-1}$
VA: $x = 1$ **Oblique asymptote:** $y = x + 3$

Graph Transformations

The Four Fundamental Transformations

Given $y = f(x)$, the following transformations produce new graphs:

Transformation	Equation	Effect on Graph
a **times** y **(vertical stretch)**	$y = af(x)$	Multiply all y-coordinates by a $\lvert a\rvert > 1$: stretch; $0 < \lvert a\rvert < 1$: compress $a < 0$: also reflects in x-axis
Add to y **(vertical shift)**	$y = f(x) + a$	Shift graph up by a units
Add to x **(horizontal shift)**	$y = f(x + a)$	Shift graph **left** by a units ("$+a$ means left" — counter-intuitive!)
a **times** x **(horizontal stretch)**	$y = f(ax)$	Multiply all x-coordinates by $\frac{1}{a}$ $\lvert a\rvert > 1$: compress horizontally $0 < \lvert a\rvert < 1$: stretch horizontally

Warning The horizontal shift "opposite" rule

$y = f(x + 2)$ shifts the graph **left** by 2, not right.
$y = f(x - 3)$ shifts the graph **right** by 3.
Think: to get the same y-value, x must be 2 units *smaller* in $f(x + 2)$, so the graph moves left.

Combining Transformations

When multiple transformations are applied, the **order matters**. The convention (and what examiners expect):

Definition Standard Order of Transformations

For $y = af(bx + c) + d$:

1. Horizontal transformations first (inside the bracket):
   - $y = f(x + c)$ — shift
   - $y = f(bx)$ — stretch/compress
2. Then vertical transformations (outside the bracket):
   - $y = af(x)$ — stretch/compress
   - $y = f(x) + d$ — shift

More precisely, the transformation from $f(x)$ to $af(bx + c) + d$ is:

$$x \mapsto \frac{x - c}{b}, \qquad y \mapsto \frac{y - d}{a}$$

Example Combined transformations

Describe the transformations mapping $y = f(x)$ to $y = 3f(2x - 4) + 1$.
Solution: Rewrite as $y = 3f(2(x - 2)) + 1$. Then:

1. Shift right by 2: $f(x - 2)$
2. Compress horizontally by factor $\frac{1}{2}$: $f(2(x - 2))$
3. Stretch vertically by factor 3: $3f(2(x - 2))$
4. Shift up by 1: $3f(2(x - 2)) + 1$

Or using the mapping formula:

$$\text{Point } (x, y) \text{ on } y = f(x) \;\mapsto\; \left(\frac{x+2}{2}, 3y + 1\right) \text{ on the new graph}$$

Warning Common ordering mistake

For $y = f(2x - 4)$, students often write: "shift left 4, then compress by factor $\frac{1}{2}$" — this is **wrong**. You must factor first: $f(2x - 4) = f(2(x - 2))$, so the shift is **right by 2** (not 4), then compress.

Special Transformations: Modulus and Reciprocal

$y = |f(x)|$

The modulus "flips" any part of $f(x)$ that is below the x-axis to above it.
Rule: Where $f(x) < 0$, reflect that portion in the x-axis. Where $f(x) \geq 0$, leave unchanged.

$y = f(|x|)$

Discard the left half of $f(x)$ (for $x < 0$), then reflect the right half ($x \geq 0$) across the y-axis.
Result: The graph is always symmetric about the y-axis (it's an even function).

Example Modulus transformations

Given $f(x) = x^2 - 4$:

- $y = |x^2 - 4|$: The part between $x = -2$ and $x = 2$ (where $f(x) < 0$) flips above the axis. Result looks like a "W" touching the x-axis at $x = \pm 2$.
- $y = f(|x|) = |x|^2 - 4 = x^2 - 4$: Since $x^2 = |x|^2$, this gives the same graph as $f(x)$ — it was already even.

$y = \frac{1}{f(x)}$

This is the reciprocal transformation. Key behaviours:

- Where $f(x) = 0$: $\frac{1}{f(x)}$ has a **vertical asymptote**
- Where $f(x) \to \pm\infty$: $\frac{1}{f(x)} \to 0$
- Where $f(x) = 1$ or $f(x) = -1$: the graphs intersect (invariant points)
- $f(x) > 0 \Rightarrow \frac{1}{f(x)} > 0$; $f(x) < 0 \Rightarrow \frac{1}{f(x)} < 0$ (sign preserved)
- As $f(x)$ increases, $\frac{1}{f(x)}$ decreases (and vice versa)

Parametric Equations

A **parametric equation** defines x and y separately in terms of a third variable (the **parameter**), usually t or θ.

Definition Parametric Equations

$x = g(t)$, $y = h(t)$, where t varies over some domain. Each value of t gives a point $(g(t), h(t))$ on the graph.

Finding the Cartesian Equation:

Eliminate t from the two equations to get a relationship between x and y.

Example Parametric to Cartesian

Given $x = t + 2$, $y = t^2 - 1$:

$$t = x - 2 \quad \text{(from the first equation)}$$
$$y = (x-2)^2 - 1 = x^2 - 4x + 3$$

So the graph is a parabola $y = x^2 - 4x + 3$.

Exam Tip Domain from parametric

When converting to Cartesian, note any domain restrictions from the parameter. E.g., if $x = 2\cos\theta$, then $x \in [-2, 2]$. The Cartesian equation without this restriction may be incomplete — state the restricted domain.

Common Misconceptions

Warning Asymptotes are lines, not numbers

Always write asymptotes as equations of lines:
$x = 3$ (not "3") $y = 2$ (not "2") $y = 2x - 1$ (not "$2x - 1$")
Writing just the number loses the mark.

Warning Horizontal vs Oblique

A rational function has a horizontal asymptote when degree(numerator) $\leq$ degree(denominator).
It has an oblique asymptote when degree(numerator) = degree(denominator) +1.
Don't do long division and call the quotient a "horizontal asymptote" — it's **oblique** (slanted).

Warning $y = f(|x|)$ vs $y = |f(x)|$

These are **completely different** operations:

- $y = |f(x)|$: reflect *below* the x-axis to above. The graph stays on or above the x-axis.
- $y = f(|x|)$: make the graph symmetric about the y-axis. The graph can still go below the x-axis.

Test with $f(x) = x - 2$ to see the difference clearly.

Connections to Other Topics

- **Equations & Inequalities (Ch 3):** Solving $\frac{f(x)}{g(x)} > 0$ requires understanding where each factor is positive/negative, which directly relates to graph signs and asymptotes.
- **Functions (Ch 1 of Functions topic):** Domain, range, inverse functions, and composite functions all rely on graph understanding. The horizontal line test for one-to-one functions is visual.
- **Calculus (later chapters):** Finding turning points and points of inflection uses differentiation. Curve sketching for any function draws on the graph characteristics learned here.
- **Vectors:** Parametric equations for lines in 3D use the same parametric concept.

GC Techniques

Exam Tip GC for graphing

Your GC is your best friend for graphing. For any function:

1. Enter the function in `Y=`
2. Set an appropriate window: use `ZOOM` → `ZStandard` first, then adjust
3. Use `CALC` → `zero` for x-intercepts, `value` for y-intercept
4. Use `CALC` → `minimum` / `maximum` for turning points
5. For asymptotes: use `TABLE` to see behaviour near suspected asymptotes

Important: The GC may show a "connect-the-dots" line across a vertical asymptote. Don't copy this — the actual graph has a break. Use your mathematical knowledge to draw the correct asymptote as a dashed line.